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Inviscid flow around bodies moving in weak density gradients without buoyancy effects

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We examine the inviscid flow generated around a body moving impulsively from rest with a constant velocity $\boldsymbol{U}$ in a constant density gradient, $\nabla\rho_0$, which is assumed to be weak in the sense $\epsilon = a|\nabla\rho_0|/\rho_0 \ll 1$, where a is the length scale of the body. In the absence of a density gradient ($\epsilon = 0$), the flow is irrotational and no force acts on the body. When $0 < \epsilon \ll 1$, vorticity is generated by a baroclinic torque and vortex stretching, which introduce a rotational component into the flow. The aim is to calculate both the flow around the body and the force acting on it.

When a two-dimensional body moves perpendicularly to the density gradient $\boldsymbol{U}\cdot\nabla\rho_0 = 0$, the density and velocity field are both steady in the body's frame of reference and the vorticity field decays with distance from the body. When a three-dimensional body moves perpendicularly to the density gradient, the vorticity field is regular in the main flow region, $\mathscr{D}_M$, but is singular in a thin inner region $\mathscr{D}_I$ located adjacent to the body and to the downstream-attached streamline, and the flow is characterized by trailing horseshoe vortices. When the body moves parallel to the density gradient $\boldsymbol{U}\times\nabla\rho_0 = \boldsymbol{0}$, the density field is unsteady in the body's frame of reference; however to leading order the flow is steady in the region $\mathscr{D}_M$ moving with the body for $Ut/a \gg 1$. In the thin region $\mathscr{D}_I$ of thickness $O(a\epsilon)$, the density gradient and vorticity are singular. When $\boldsymbol{U}\times\nabla\rho_0 = \boldsymbol{0}$ this singularity leads to a downstream 'jet' with velocities of $O(-(\boldsymbol{U}\cdot\nabla\rho_0)Ua/(\rho_0 U))$ on the downstream attached streamline(s). In the far field the flow is characterized by a sink of strength $C_M\mathscr{V}(\boldsymbol{U}\cdot\nabla\rho_0)/2\rho_0$, located at the origin, where C_M is the added-mass coefficient of the body and $\mathscr{V}$ is the body's volume.

The forces acting on a body moving steadily in a weak density gradient are calculated by considering the steady relative velocity field in region $\mathscr{D}_M$ and evaluating the momentum flux far from the body. When $\boldsymbol{U}\cdot\nabla\rho_0 = 0$, a lift force, $C_L\mathscr{V}(\boldsymbol{U}\times\nabla\rho_0)\times\boldsymbol{U}$, pushes the body towards the denser fluid, where the lift coefficient is $C_L = C_M/2$ for a three-dimensional body, that is axisymmetric about $\boldsymbol{U}$, and is $C_L = (C_M+1)/2$ for a two-dimensional body. The direction of the lift force is unchanged when $\boldsymbol{U}$ is reversed. A general expression for the forces on bodies moving in a weak shear and perpendicularly to a density gradient is calculated. When $\boldsymbol{U}\times\nabla\rho_0 = \boldsymbol{0}$, a drag force $-C_D\mathscr{V}(\boldsymbol{U}\cdot\nabla\rho_0)\boldsymbol{U}$ retards the body as it moves into denser fluid, where the drag coefficient is $C_D = C_M/2$, for both two- and three-dimensional axisymmetric bodies. The direction of the drag force changes sign when $\boldsymbol{U}$ is reversed. There are two contributions to the drag calculation from the far field; the first is from the wake 'jet' on the attached streamline(s) caused by the rotational component of the flow and this leads to an accelerating force. The second and larger contribution arises from a downstream density variation, caused by the distortion of the isopycnal surfaces by the primary irrotational flow, and this leads to a drag force.

When cylinders or spheres move with a velocity $\boldsymbol{U}$ at arbitrary orientation to the density gradient, it is shown that they are acted on by a linear combination of lift and drag forces. Calculations of their trajectories show that they initially slow down or accelerate on a length scale of order $\rho_0/|\nabla\rho_0|$ (independent of $\mathscr{V}$ and $\boldsymbol{U}$) as they move into regions of increasing or decreasing density, but in general they turn and ultimately move parallel to the density gradient in the direction of increasing density gradient.

1. Introduction

Particle-laden flows, high-speed low-density jets surrounded by high-density fluid, and turbulent flames are examples of flows where solid particles or lumps of fluid with different densities move in complex flow fields with density gradients. In these situations the effect of density gradients on the flows and on the pressure gradients may be greater than those caused by buoyancy forces, and can alter the eddy structure and entrainment into plumes (e.g. Rooney & Linden 1996) and shear layers (Hermanson & Dimotakis 1989). Previous studies of two-phase flows have focused on how the flow and forces on particles depend on external factors such as the gradient of the local velocity field (Auton, Hunt & Prud'homme 1988) and on differences between the density of the local fluid and that of the body. Current models of turbulent two-phase flows allow for the inertial effects on the movement of eddies (or more accurately control volumes) having a different density from their surroundings (Hunt, Perkins & Fung 1995), but do not generally represent the effect of the local gradients of density on the forces acting on the particles or eddies. Although the effect of a mean density gradient on turbulence is still not fully understood, Chassaing, Harran & Joly (1994) and Panchapakesan & Lumley (1993) have made progress in understanding the effect of density fluctuations on the mean flow of variable-density jets.

In this paper we examine the flow generated by a body moving with constant velocity $\boldsymbol{U}$ in a constant density gradient, $\nabla\rho_0$, with the aim of understanding and estimating the forces and movement of eddies or particulate in flows with variable density. We focus on flows characterized by high Reynolds and Froude numbers where inertia forces dominate over viscous and buoyancy forces. These problems have not previously been studied in detail, especially when the body moves parallel to the density gradient.

The inviscid flow around a sphere moving perpendicularly to a gradient of density ($\boldsymbol{U}.\nabla\rho_0 = 0$) was examined by Hawthorne & Martin (1955)† who showed that trailing horseshoe vortices are generated by a baroclinic torque and vortex stretching. Their flow prediction was confirmed qualitatively in wind-tunnel flow visualization experiments. Drazin (1961) calculated the vorticity distribution around bodies moving perpendicularly to the density gradient, but did not calculate the detailed flow around such bodies. Neither paper included calculations of the complete velocity field nor the forces which act on the body. The flow generated by a body moving parallel to the density gradient has been calculated only when buoyancy effects are present. For instance, Zvirin & Chadwick (1975) showed that when the flow is characterized by a low Reynolds number, there is an additional drag force tending to retard the body as it moves into more dense/lighter fluid. Warren (1960) calculated the drag force on a

† There is a sign error in (19) of Hawthorne & Martin (1955), but otherwise the interpretation of the experiments is, we believe, correct.

slender body moving parallel to the density gradient and to gravitational acceleration in an inviscid Boussinesq fluid and likewise showed that a drag force acts to retard the body as it moves into denser/lighter fluid.

Although there has been no complete solution for the inviscid velocity field as the body moves perpendicularly to the density gradient, it is possible to use the transformation proposed by Yih (1959) to relate the results of previous research on the flow around bodies moving steadily in sheared flows to the flow around bodies moving perpendicularly to the density gradient. Yih (1959) showed that the steady velocity field $\boldsymbol{v}$ (in a frame moving with the body of speed $\boldsymbol{U}$) in the former problem is equivalent to the rescaled velocity field in the latter problem through the relation,

$$\boldsymbol{v}' = \left(\rho/\rho_B\right)^{1/2}\boldsymbol{v}, \tag{1.1}$$

where $\boldsymbol{v}' \to -(\rho_0(y)/\rho_B)^{1/2}\boldsymbol{U}$ far from the body, $\rho_0(y)$ is the unperturbed density field and $\rho_B = \rho_0(0)$. No linearization is involved and this transformation is valid in two and three dimensions (see Appendix A).

We make use of the work of Taylor (1917) and Batchelor (1967) for the flow generated by two-dimensional bodies moving in a shear, even when the shear is large. Lighthill (1956) examined the flow generated by a sphere moving in a weak shear, which Auton (1987) used to calculate the lift force acting on the sphere. Trailing horseshoe vortices are generated which tend to push the sphere towards the faster-moving fluid. A feature of these and other previous calculations of the effects of upstream shear and/or density gradient is that the vorticity is singular on the surface of three-dimensional bodies and on downstream-attached streamlines where the drift function is singular. Some of the significant implications, which were not worked out there, are examined in this paper. Wallis (1996) generalized Auton's lift force calculation to the case of three-dimensional bodies and verified the value of the lift coefficient experimentally for a number of axisymmetric bodies.

Most previous studies of the effect of density gradient have focused on buoyancy forces where gravity is aligned with $\nabla\rho_0$ or any flows where the body or flow moves perpendicularly to the density gradient (see Turner 1973). There are a few studies of the flow generated by a body moving parallel to the density gradient (e.g. Warren 1960; Zvirin & Chadwick 1975); however, these studies do not particularly help to solve the problem when buoyancy effects are negligible. The aim of this paper is to examine the inviscid incompressible flow around a rigid body moving in a weak density gradient in the absence of buoyancy effects. The parameter describing the strength of the unperturbed density gradient is $\epsilon = |\nabla\rho_0|a/\rho_0 \ll 1$, where a is the characteristic body length scale. In the absence of the density gradient ($\epsilon = 0$), the flow is irrotational and no force acts on the body. When $0 < \epsilon \ll 1$, the density gradients generate vorticity of $O(\epsilon|\boldsymbol{U}|/a)$ through the actions of the baroclinic torque and vortex stretching. The rotational component of the flow changes the pressure distribution on the body surface which leads to a force, $\boldsymbol{F}$, acting on the body. Our aim is to calculate both the flow generated by the body and the force acting on it.

The problem we examine in this paper, of a body moving in a relatively weak non-uniform density field, is defined mathematically in §2. The perturbation density field caused by the irrotational component of the flow around a body travelling through a uniform density gradient is calculated in §3. The flow around a body moving perpendicularly or parallel to the density gradient is calculated in §4 and §5 respectively. The force acting on a body moving parallel and perpendicular to the density gradient are evaluated using the momentum integral approach in §6. The total force acting on a cylinder or sphere projected into a weak density gradient is calculated in §7 and is used to predict the trajectory of the body.

2. Problem definition

Formally, our problem is to analyse the incompressible inviscid flow around a rigid body moving impulsively from rest at $t = 0$ with a constant velocity $\boldsymbol{U} = (U, 0, 0)$ into a region denoted by $\mathscr{D}$, where initially the flow is at rest and the density field, $\rho_0(\boldsymbol{x})$, has a uniform gradient, $\nabla\rho_0$. The solutions to the velocity and density fields $\boldsymbol{u}(\boldsymbol{x}, t)$, $\rho(\boldsymbol{x}, t)$ satisfy the incompressibility condition

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0, \tag{2.1a}$$

the continuity equation

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.1b}$$

and the conservation of momentum

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p. \tag{2.1c}$$

Clearly, the Boussinesq approximation, where density changes in the inertia term are negligible, is inapplicable. In order to define a suitable coordinate system, $\boldsymbol{x}_B$ is taken as a reference point in the body which is initially at $\boldsymbol{x}_0$, as shown in figure 1. Then for $t > 0$, the position of the body is $\boldsymbol{x}_B(t) = \boldsymbol{x}_0 + \boldsymbol{U}t$. We introduce the coordinate $\boldsymbol{x}'$ relative to the body, $\boldsymbol{x} = \boldsymbol{x}_B(t) + \boldsymbol{x}'$, and a new velocity field $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{U}$ relative to the body. Now the velocity and density fields are subject to the boundary conditions: far upstream of the body, as $\boldsymbol{x}' \cdot \boldsymbol{U} \to \infty$,

$$\boldsymbol{u} \to \boldsymbol{0} \quad \text{or} \quad \boldsymbol{v} \to -\boldsymbol{U}, \tag{2.2}$$

and

$$\rho \to \rho_0(\boldsymbol{x}). \tag{2.3}$$

On the surface S_B of a rigid body (defined as the points $\boldsymbol{x}'_S$ which are the solutions of $f_S(\boldsymbol{x}) = 0$), the kinematic condition satisfied by the velocity field is

$$\boldsymbol{v} \cdot \hat{\boldsymbol{n}} = 0, \tag{2.4}$$

where $\hat{\boldsymbol{n}} = \nabla f_S / |\nabla f_S|$ is directed into the body.

The aim of this paper is to calculate the flow generated by the body and the force $\boldsymbol{F}$ (defined in §6) which acts on the body. We consider a body moving in a fluid with a constant density gradient, and write the velocity and density fields as reference fields and perturbations, namely

$$\boldsymbol{v}(\boldsymbol{x}', t) = \boldsymbol{v}_1(\boldsymbol{x}') + \boldsymbol{v}_2(\boldsymbol{x}', t), \tag{2.5}$$

$$\rho(\boldsymbol{x}', t) = \rho_1(\boldsymbol{x}', t) + \rho_2(\boldsymbol{x}', t), \tag{2.6}$$

and

$$\rho_0(\boldsymbol{x}) = \rho_0(\boldsymbol{0}) + (\boldsymbol{x} - \boldsymbol{x}_0) \cdot \nabla\rho_0, \quad \text{where} \quad \text{w.l.o.g.} \quad \boldsymbol{x}_0 = \boldsymbol{0}. \tag{2.7}$$

Here the velocity field, $\boldsymbol{v}_1$, represents the irrotational flow in the absence of a density gradient, and ρ_1 represents the density field perturbed by the irrotational flow. The problem is unsteady because $\boldsymbol{x}'$ depends on time. However, we shall demonstrate that in a moving frame to leading order there is a steady solution defined in a region $\mathscr{D}_M \subset \mathscr{D}$. The excluded inner region $\mathscr{D}_I$ is shown to be much smaller than $\mathscr{D}_M$. We will show that the residual terms, $\boldsymbol{v}_2$ and ρ_2, are relatively small in $\mathscr{D}_M$.

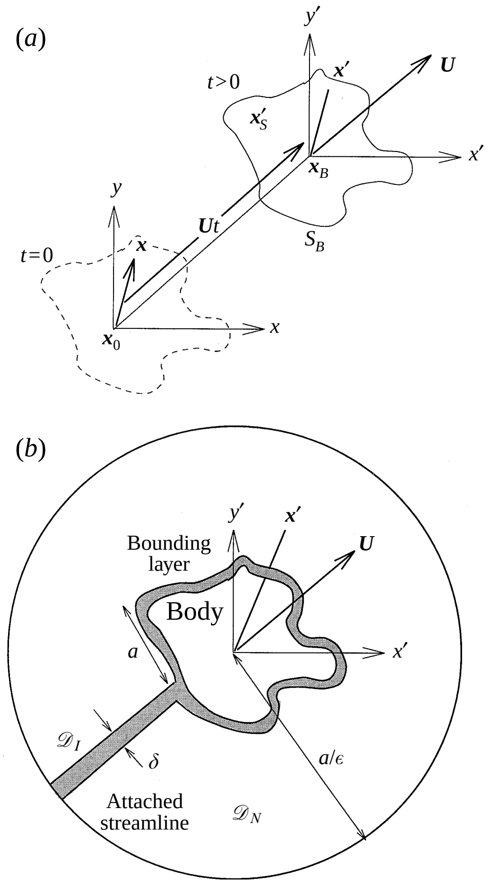

FIGURE 1. Schematic defining the coordinate system is shown in (*a*). The inner region $\mathscr{D}_I$ and main region $\mathscr{D}_M$ are sketched in (*b*).

In the absence of a density gradient, from (2.1*c*) the fluid is irrotational and steady in the frame moving with the body, i.e.

$$\boldsymbol{v}_1(\boldsymbol{x}, t) = \boldsymbol{v}_1(\boldsymbol{x}') = \nabla\phi_1 \quad \text{for} \quad t > 0, \tag{2.8}$$

where from (2.1*b*), (2.2), (2.3) the velocity potential, ϕ_1, satisfies Laplace's equation ($\nabla^2\phi_1 = 0$), the boundary condition

$$\nabla\phi_1 \cdot \hat{\boldsymbol{n}} = 0, \quad \text{on} \quad S_B \tag{2.9}$$

and the far field condition $\nabla\phi_1 \to -\boldsymbol{U}$ as $|\boldsymbol{x}'| \to \infty$.

In the presence of a density gradient, a secondary velocity field $\boldsymbol{v}_2$ is generated which satisfies $\boldsymbol{v}_2 \cdot \hat{\boldsymbol{n}} = 0$ on S_B. By definition, the density field, $\rho_1(\boldsymbol{x}, t)$, is determined by $\boldsymbol{v}_1$, so from (2.1*c*)

$$\frac{\partial\rho_1}{\partial t} + \boldsymbol{v}_1 \cdot \nabla\rho_1 = 0, \tag{2.10}$$

and ρ_1 is subject to the boundary condition that the perturbed density gradient tends to the undisturbed value $\nabla\rho_0$ far upstream of body, namely as $\boldsymbol{x}' \cdot \boldsymbol{U} \to \infty$

$$\rho_1(\boldsymbol{x}, t) \to \rho_0(\boldsymbol{0}) + \boldsymbol{x} \cdot \nabla\rho_0 = \rho_B + \boldsymbol{x}' \cdot \nabla\rho_0 \quad \text{and thence} \quad (\partial\nabla\rho_1/\partial t)_{\boldsymbol{x}'=const} \to 0, \tag{2.11a}$$

where $\rho_B = \rho_0(\boldsymbol{0}) + \boldsymbol{U}t \cdot \nabla\rho_0$ is the density at $\boldsymbol{x}_B$ in the absence of the body. Since

$\boldsymbol{v}_1(\boldsymbol{x}')$ is independent of t, (2.11) becomes in the frame of the body

$$\nabla(\boldsymbol{v}_1 \cdot \nabla\rho_1) = \boldsymbol{0} \quad \text{or} \quad \boldsymbol{v}_1 \cdot \nabla\rho_1 = -\boldsymbol{U} \cdot \nabla\rho_0. \tag{2.11b}$$

The perturbation component, ρ_2, must satisfy

$$\frac{\mathrm{D}\rho_2}{\mathrm{D}t} + \boldsymbol{v}_2 \cdot \nabla\rho_1 = 0, \tag{2.12}$$

subject to the boundary condition $\rho_2 \to 0$ as $\boldsymbol{x}'.\boldsymbol{U} \to \infty$. Since $\boldsymbol{v}_1$ is irrotational the vorticity, $\nabla \times \boldsymbol{v} = \nabla \times \boldsymbol{v}_2 = \boldsymbol{\omega}_2$, is generated by the action of the baroclinic torque on the flow. The vorticity equation is obtained by taking the curl of (2.1*a*):

$$\frac{\mathrm{D}(\rho\boldsymbol{\omega}_2)}{\mathrm{D}t} = \frac{\mathrm{D}\boldsymbol{v}_1}{\mathrm{D}t} \times \nabla\rho_1 + (\rho\boldsymbol{\omega}_2 \cdot \nabla)\boldsymbol{v}_1 + \frac{\mathrm{D}\boldsymbol{v}_2}{\mathrm{D}t} \times \nabla\rho_1 + \frac{\mathrm{D}\boldsymbol{v}_1}{\mathrm{D}t} \times \nabla\rho_2 + \frac{\mathrm{D}\boldsymbol{v}_2}{\mathrm{D}t} \times \nabla\rho_2 + (\rho\boldsymbol{\omega}_2 \cdot \nabla)\boldsymbol{v}_2. \tag{2.13}$$

When the body moves perpendicularly to the density gradient $\boldsymbol{U} \cdot \nabla\rho_0 = 0$, the streamwise component of vorticity can be calculated exactly from the above equation (Scorer 1978), to give

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_2 \cdot \hat{\boldsymbol{s}}}{v}\right) = \frac{2\kappa}{\rho v}\nabla\left(\tfrac{1}{2}\rho|\boldsymbol{U}|^2\right) \cdot \hat{\boldsymbol{b}} \tag{2.14}$$

where $1/\kappa$ is the radius of curvature of the streamlines. The local unit vectors $\hat{\boldsymbol{b}}$, $\hat{\boldsymbol{s}}$ and $\hat{\boldsymbol{n}}$ are respectively binormal, tangential and normal to the streamlines (towards the centre of curvature) and satisfy the relationship $\hat{\boldsymbol{s}} \times \hat{\boldsymbol{n}} = \hat{\boldsymbol{b}}$. Linearizing (2.15) shows that the streamwise component of vorticity is

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_2 \cdot \hat{\boldsymbol{s}}}{v_1}\right) = \frac{1}{\rho v_1^3}\frac{\partial v_1^2}{\partial n}\frac{\partial(\frac{1}{2}\rho_1|\boldsymbol{U}|^2)}{\partial b}. \tag{2.15}$$

This equation implies that for three-dimensional bodies (where $\partial/\partial b \neq 0$), the streamwise component of vorticity persists far downstream of the body – this is described as 'trailing' vorticity.

When the body moves perpendicularly to the density gradient, ($\boldsymbol{U} \cdot \nabla\rho_0 = 0$), the density field $\rho_1(\boldsymbol{x}')$ is steady because the isopycnal surfaces are not permanently displaced forward and is weak in the sense that its gradient is $O(|\nabla\rho_0|)$. However, in three-dimensional flows, vortex stretching generates a vorticity field which grows without limit within a thin layer located on the surface of the body S_B and surrounding the attached downstream streamlines – this layer is termed region $\mathscr{D}_I$ (see figure 1*b*). Despite the singularity of the vorticity distribution, the velocity field is steady everywhere in $\mathscr{D}$, i.e. both in the inner region $\mathscr{D}_I$ and in the main region $\mathscr{D}_M$. The perturbed velocity field is $|\boldsymbol{v}_2| = O(|\boldsymbol{U}|\epsilon)$ and so from (2.13) the density field $\rho_2 = O(\rho_0\epsilon^2)$. The formal link between the flow generated by a body moving perpendicularly to the density gradient and parallel to a shear flow is explained in Appendix A. We note that both have this feature of a singularity in vorticity in region $\mathscr{D}_I$.

When the body moves parallel to the density gradient, $\boldsymbol{U} \times \nabla\rho_0 = \boldsymbol{0}$, the density field is unsteady in the body frame of reference because isopycnal surfaces are permanently displaced forward (see figure 2). These surfaces 'pile-up' on the body and are stretched as they are convected downstream resulting in a density gradient which is singular on the surface of the body and attached streamline(s). The gradient is singular because the isopycnal surfaces are passively advected close to a stagnation point. When the bodies do not possess stagnation points (e.g. bodies which are cusped at the points of attached streamlines) and/or when molecular diffusion is

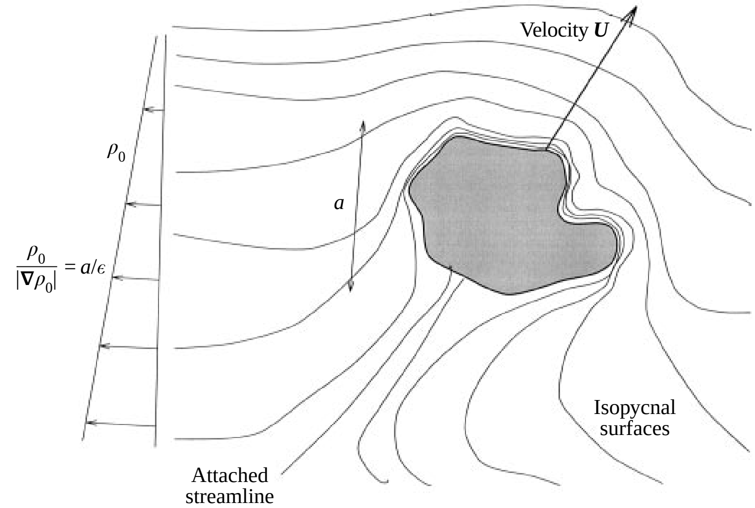

FIGURE 2. Schematic of the distortion of isopycnal surfaces by a body moving in a variable-density flow, showing the different length scales of the body and density gradient.

present, the density gradients are not singular and region $\mathscr{D}_I$ is not present. The density gradient is unsteady within a thin region $\delta = O(a\exp(-|\boldsymbol{U}|t/a))$, which decreases in time – this width defines the region $\mathscr{D}_I$, and when $|\boldsymbol{U}|t/a \gg 1$ the density gradient $\nabla\rho_1$ is steady (from 2.12*a*) in $\mathscr{D}_M$. In region $\mathscr{D}_M$, where $|\nabla\rho_1|/\rho_0 \ll 1/a$, $|\boldsymbol{v}_2| \ll |\boldsymbol{v}_1 + \boldsymbol{U}|$, $|\nabla\rho_2| \ll |\nabla\rho_1|$, the vorticity is small compared to the irrotational strain and is steady. However, in the thin region $\mathscr{D}_I$ adjacent to the surface of the body and the attached streamline(s) which do not come from upstream, the density gradients are large enough that the perturbation velocity $|\boldsymbol{v}_2|$ is comparable to $|\boldsymbol{v}_1 + \boldsymbol{U}|$. Because ρ_1 and $\boldsymbol{v}_2$ have weak singularities in $\mathscr{D}_I$, the mass and volume flux averaged over the region are small. The associated nonlinear unsteady effects are neglected from our analysis because they are confined to a narrow region, $\mathscr{D}_I$, whose effect on the overall flow and the force on the body is negligible to leading order.

From (2.14), it is clear that the existence of trailing vorticity induces a secondary velocity field $\boldsymbol{v}_2$ which decays more slowly with distance from the body than $(\boldsymbol{v}_1 + \boldsymbol{U})$. As a result, when $r/a = O(\epsilon^{-1})$, the velocity gradients of the secondary flow $(\boldsymbol{v}_2)$ dominate the strain field far from the body because they are not small compared with those of the primary flow. Therefore the linearized solution is not valid for $r > a/\epsilon$. However, when we evaluate the force on the body, we choose a control surface which is smaller than a/ϵ, where the 'near field' solution $(\boldsymbol{v}_1 + \boldsymbol{U})$ dominates $\boldsymbol{v}_2$.

As $\epsilon \to 0$, the vorticity equation (2.14) tends to

$$\frac{\mathrm{D}'(\rho_1\boldsymbol{\omega}_2)}{\mathrm{D}'t} = \frac{\mathrm{D}'\boldsymbol{v}_1}{\mathrm{D}'t} \times \nabla\rho_1 + (\rho_1\boldsymbol{\omega}_2 \cdot \nabla)\boldsymbol{v}_1, \tag{2.16}$$

where $\mathrm{D}'/\mathrm{D}'t = \partial/\partial t + \boldsymbol{v}_1.\nabla$ and the density field, ρ_1, satisfies (2.12*b*). The secondary velocity field can be written as, $\boldsymbol{v}_2 = \boldsymbol{v}_2^r + \nabla\phi_2$, where $\boldsymbol{v}_2^r$ and $\nabla\phi_2$ are respectively the rotational and irrotational components of the secondary flow. Formally we seek to solve for the unknowns $\boldsymbol{v}_2^r$ and ρ_1 subject to (2.17), (2.12*b*), to the kinematic conditions,

$$\nabla \times \boldsymbol{v}_2^r = \boldsymbol{\omega}_2, \quad \nabla^2\phi_2 = 0, \tag{2.17}$$

and to the boundary conditions

$$(\boldsymbol{v}_2^r + \nabla\phi_2)\cdot\boldsymbol{n} = 0 \quad \text{on} \quad S_B, \tag{2.18}$$

$$(\boldsymbol{v}_2^r + \nabla\phi_2) \to \boldsymbol{0} \quad \text{as} \quad \boldsymbol{x}'\cdot\boldsymbol{U} \to \infty, \tag{2.19}$$

and

$$\rho_1(\boldsymbol{x}') \to \rho_0(\boldsymbol{x}') \quad \text{as} \quad \boldsymbol{x}'\cdot\boldsymbol{U} \to \infty. \tag{2.20}$$

We proceed by first calculating the perturbed density field, $\rho_1(\boldsymbol{x}')$. Throughout this paper, the body moves parallel to the x-axis, so $\boldsymbol{U} = (U, 0, 0)$, and the 'base streamlines' are those of the unperturbed irrotational flow.

3. Density field perturbed by the irrotational flow, ρ_1

The perturbed density field, ρ_1, satisfying (2.12b) subject to the boundary condition (2.12a), can be calculated exactly by finding the mapping between the Lagrangian and Eulerian frames of reference. Since we assume that the unperturbed density field is linear in $\boldsymbol{x}$, after the fluid particles have been displaced to points (X, Y), the density field can be written in terms of the Lagrangian coordinates

$$\rho_0(X, Y) = \rho_0(\boldsymbol{0}) + X\frac{\mathrm{d}\rho_0}{\mathrm{d}x} + Y\frac{\mathrm{d}\rho_0}{\mathrm{d}y}, \tag{3.1}$$

where X, Y are the Lagrangian coordinates. Figure 3 shows a comparison between the density field in the Lagrangian and Eulerian frames of reference for a cylinder. The dashed lines show the isopycnal surfaces associated with a vertical density gradient, and the full lines are associated with a horizontal density gradient. The Lagrangian and Eulerian coordinates (x', y') are related by

$$(X, Y) = (x', y') - \int_0^t (v_{1,x}, v_{1,y})\mathrm{d}t. \tag{3.2}$$

The solution to (3.1) is

$$\rho_1(x', y', z') = \rho_0(X(x', y', z'), Y(x', y', z')) \tag{3.3}$$

(Hunter 1983, p. 43). When the flow is irrotational, $\boldsymbol{v}_1 = \nabla\phi_1$, and $\boldsymbol{v}_1 + \boldsymbol{U} \to \boldsymbol{0}$ as $|\boldsymbol{x}'| \to \infty$; also Eames, Belcher & Hunt (1994) showed that the streamwise (or x-direction) displacement of a fluid particle is

$$\int_0^t v_{1,x}\mathrm{d}t = -\frac{\phi_1}{U} - x' + X_d - Ut,$$

where we define Darwin's 'displacement' drift function, X_d, to be

$$X_d(\boldsymbol{x}') = \int_0^t \frac{|\nabla\phi_1 + \boldsymbol{U}|^2}{U}\mathrm{d}t.$$

Therefore the horizontal Lagrangian coordinate is

$$X(\boldsymbol{x}') = 2x' + \frac{\phi_1}{U} - X_d + Ut. \tag{3.4}$$

Since Darwin's drift function X_d depends on the velocity perturbation, it increases along a streamline, and decays rapidly from the centreline. Downstream, Darwin's displacement function, $\lim_{x'\to-\infty} X_d(\boldsymbol{x}')$ describes the permanent displacement of a material surface by a body for which asymptotic expressions have been calculate for

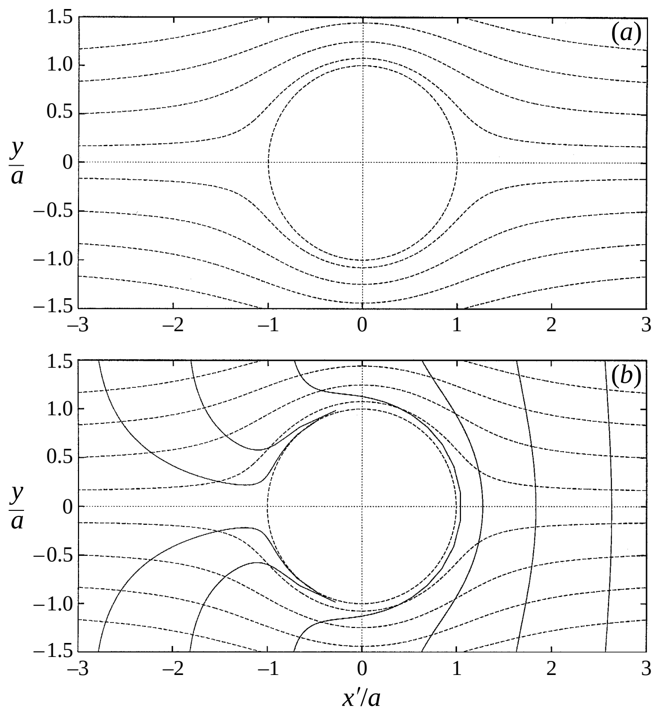

FIGURE 3. The relationship between the Lagrangian and Eulerian frames of reference, using for illustration the flow around a cylinder of radius a. The streamlines describing the flow around the cylinder are plotted in (a). The isopycnal surfaces are deformed by the flow around the cylinder and plotted in (b). The surfaces initially parallel to the x-axis are deflected around the body and coincide with the streamlines. The isopycnal surfaces initially parallel to the y-axis accumulate on the surface of the body.

a sphere (Lighthill 1956) and cylinder (Darwin 1953): e.g. for a cylinder

$$\lim_{x'\to-\infty} X_d(\boldsymbol{x}') \sim \begin{cases} \dfrac{\pi a^4}{2y^3}, & y \gg a, \\ a\left(\log\left(\dfrac{8a}{y}\right) - 2\right), & y \ll a. \end{cases}$$

The downstream displacement is singular close to the centreline because there the fluid has passed close to the stagnation points. This singularity is not present when the bodies are cusped at the attached streamlines. The volume of fluid permanently displaced forward by the passage of a body is the integral $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \lim_{x'\to-\infty} X_d \mathrm{d}y'\mathrm{d}z'$, which is equal to $C_M\mathscr{V}$, where C_M is the added-mass coefficient of the body (Darwin 1953). The added-mass coefficient is a geometrical factor which depends on the relative orientation of the body to the flow and is defined by

$$C_M\mathscr{V} = \frac{\boldsymbol{U}^T\boldsymbol{\alpha}\boldsymbol{U}}{U^2}, \tag{3.5}$$

where $\boldsymbol{\alpha}$ is the added-mass tensor (Batchelor 1967, p. 403). The added-mass coefficients for a sphere ($C_M = 1/2$) and cylinder ($C_M = 1$) are independent of the relative orientation to the flow because these bodies have point symmetry.

To calculate the vorticity generated by the baroclinic torque, we use Lighthill's (1956) 'time' drift function, τ, defined by

$$\tau = \int_{-\infty}^{\phi_1} \frac{\mathrm{d}\phi_1}{v_1^2}. \tag{3.6}$$

Lighthill's drift function describes the time taken for a fluid particle to be advected between two points on a streamline and is related to the horizontal displacement, X, through

$$U\tau = \int_{-\infty}^{\phi_1} \frac{U}{v_1^2}\mathrm{d}\phi_1 = \int_{-\infty}^{\phi_1} \frac{|\nabla\phi_1 + \boldsymbol{U}|^2 - 2v_{1,x}U - v_1^2)}{Uv_1^2}\mathrm{d}\phi_1 = X_d - 2x' - \frac{\phi_1}{U} = Ut - X(\boldsymbol{x}'). \tag{3.7}$$

The cross-stream Lagrangian coordinate, Y, is constant along a streamline and only depends on the height of the streamline from the plane $y = 0$, far up/downstream of the body. Thus Y is determined by the streamfunction: e.g. in two-dimensional flow,

$$Y = -\psi_1/U, \tag{3.8}$$

where ψ_1 is the streamfunction corresponding to the velocity potential ϕ_1. In a three-dimensional axisymmetric flow the streamlines lie in a fixed plane $z/y = \cos\alpha$ and the height of the streamline from the plane $y = 0$ far up/downstream of the body is

$$Y = (-2\psi_1/U)^{1/2}\cos\alpha, \tag{3.9}$$

where the axisymmetric flow is characterised by ψ_1, the Stokes streamfunction. Combining (3.4), (3.8) and (3.9) shows that the density field caused by the movement of a two-dimensional body is

$$\rho_1(\boldsymbol{x}') = \rho_B + \left(2x' - \frac{\phi_1}{U} - X_d\right)\frac{\mathrm{d}\rho_0}{\mathrm{d}x} - \frac{\psi_1}{U}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}, \tag{3.10}$$

and of an axisymmetric body is

$$\rho_1(\boldsymbol{x}') = \rho_B + \left(2x' - \frac{\phi_1}{U} - X_d\right)\frac{\mathrm{d}\rho_0}{\mathrm{d}x} + \left(\frac{-2\psi_1}{U}\right)^{1/2}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}\cos\alpha, \tag{3.11}$$

where $\rho_B = \rho_0(\boldsymbol{0}) + \boldsymbol{U}t\cdot\nabla\rho_0$.

4. Flow around a body moving perpendicularly to the density gradient, $\boldsymbol{U}\cdot\nabla\rho_0 = 0$

4.1. *Two-dimensional analysis*

The vorticity field can be calculated from (2.17), in terms of the Lagrangian coordinates

$$\boldsymbol{\omega}_2(\boldsymbol{x}')\cdot\hat{\boldsymbol{z}} = -\frac{1}{2U\rho_0(Y)}\frac{\mathrm{d}\rho_0(Y)}{\mathrm{d}y}\left(\boldsymbol{v}_1^2 - U^2\right). \tag{4.1}$$

This solution (Drazin 1961) is valid for any density field $\rho_0(Y)$, but the magnitude of $\mathrm{d}\rho_0/\mathrm{d}y$ is restricted because the secondary vorticity is assumed to be weak (i.e. $\epsilon \ll 1$). The vorticity field is localized, and decays away from the body. The flow is calculated by writing the secondary velocity $\boldsymbol{v}_2^r$ as $\boldsymbol{v}_2^r = \nabla\times\psi_2^r\hat{\boldsymbol{z}}$ and solving (2.18) and (4.1).

The primary flow is dipolar in the far field so that

$$\phi_1(\boldsymbol{x}') = -Ur\cos\theta + \mu U\frac{\cos\theta}{r} \quad \text{as} \quad r/\mu^{1/2}\to\infty, \tag{4.2}$$

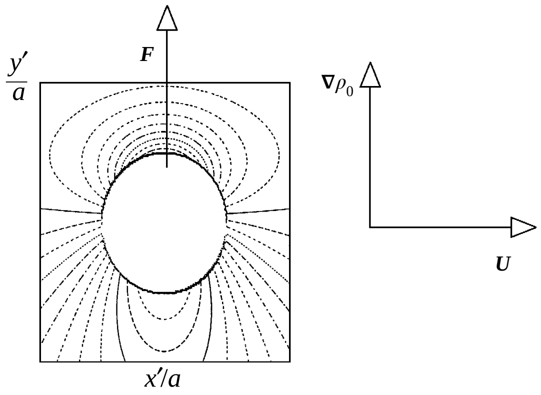

FIGURE 4. Streamlines describing the flow around a cylinder in the fluid frame of reference. The cylinder is moving perpendicularly to the density gradient and the density gradient is characterized by $\epsilon = 1$. A large density gradient is taken so that the perturbation to the streamlines is large and can be see. The direction of the force on the cylinder, **F**, is shown.

where the dipole strength is $\mu = (C_M + 1)\mathscr{V}/2\pi$ (Taylor 1928). Far from the body, the secondary flow calculated from (4.1) is

$$\boldsymbol{v}_2^r(\boldsymbol{x}') = -\frac{U\mu}{\rho_B}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}\left(\frac{\sin 2\theta}{2r}\hat{\boldsymbol{r}} - \frac{\mu}{2r^3}\hat{\boldsymbol{\theta}}\right) \quad \text{for} \quad 1 \ll r/\mu^{1/2} \ll 1/\epsilon. \tag{4.3}$$

The secondary flow consists of a circulatory and a rotational quadrupole term. In general, the flow is dominated by the rotational quadrupole term which tends to speed up the fluid above the body and decelerate the fluid beneath it, resulting in a pressure drop across the flow and a lift force which pushes the body towards the denser fluid. The perturbation velocity field is symmetric in the sense that its direction is reversed when the relative velocity between the body and fluid changes, so that the lift force is in the same direction when the body is travelling in the opposite direction.

The primary flow around a cylinder is described exactly by (4.2) with $\mu = a^2$, and the secondary flow may be written in terms of the streamfunction

$$\psi_2^r(r,\theta) = \frac{Ua^2}{4\rho_B}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}\left(\cos 2\theta + \frac{a^2}{2r^2}\right). \tag{4.4}$$

The irrotational component, ϕ_2, satisfying the boundary condition (2.19) is

$$\phi_2 = \frac{\Gamma}{2\pi}\theta - \frac{Ua^2}{4\rho_B}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}\frac{\sin 2\theta}{r^2}. \tag{4.5}$$

The strength of the line vortex, Γ, is indeterminate even though the flow it induces satisfies the kinematic condition on the surface of the cylinder and in the far field. However, an infinite amount of kinetic energy must be introduced to generate the flow induced by a line vortex; this is an unphysical assumption, so that $\Gamma = 0$. This could also be derived from the initial value problem for $t > 0$. Therefore, the streamfunction for the velocity field, $\boldsymbol{v}$, around the cylinder is

$$\psi = -Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta + \frac{Ua^2}{4\rho_B}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}\left\{\left(1 - \frac{a^2}{r^2}\right)\cos 2\theta + \frac{a^2}{2r^2}\right\}. \tag{4.6}$$

The last term gives the effective circulation around the body which induces the lift force. Figure 4 shows the streamlines for the full velocity field (i.e. $\boldsymbol{v} + \boldsymbol{U}$) around the

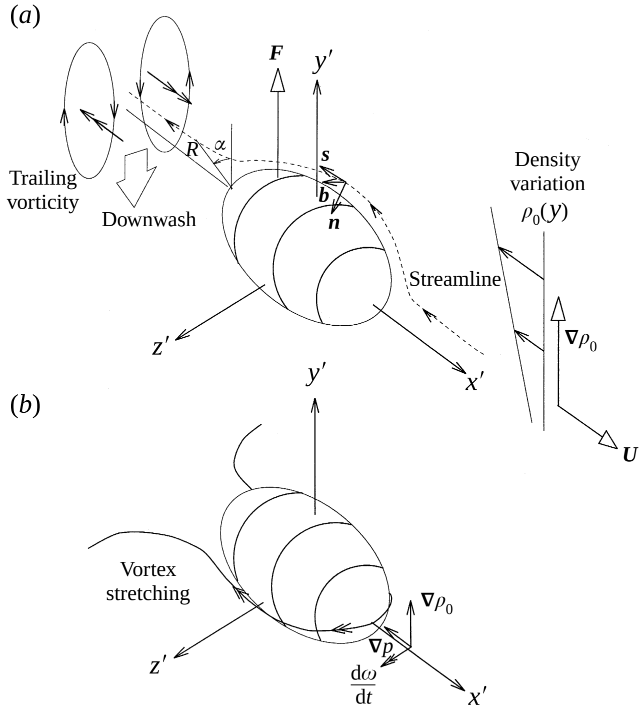

FIGURE 5. Schematic the flow past an axisymmetric body showing (*a*) the notation and the downwash produced by the baroclinic, non-buoyancy effect, and (*b*) how vorticity is generated upstream of the obstacle by the baroclinic torque produced by the unperturbed density field and how the vorticity is then distorted and amplified as it is advected around the body.

cylinder in a fixed frame of reference when $\mathrm{d}\rho_0/\mathrm{d}y > 0$. Note that the streamfunction describing the flow past a cylinder (4.6) is only valid in the region $r \ll a/\epsilon$, where $\epsilon = a|\mathrm{d}\rho_0/\mathrm{d}y|/\rho_0 \ll 1$.

However, for the specific case when the unperturbed density field has a quadratic dependence on y, so that $\rho_0(1+\epsilon y/2a)^2$ (where ϵ is not necessarily small), the streamfunction can be derived for the flow *everywhere*, namely $\psi = -2(Ua/\epsilon)\left(1-\psi_s\epsilon/Ua\right)^{1/2}$, where ψ_s is the streamfunction describing the inviscid flow around a two-dimensional body fixed in a constant shear $\epsilon U/2a$. For the specific case of a cylinder (Batchelor 1967, p. 543)

$$\psi_s = -Ur\left(1-\frac{a^2}{r^2}\right)\sin\theta + \frac{Ua\epsilon}{8}\left(-\frac{a^2}{r^2}\cos 2\theta - 2\frac{r^2}{a^2}\sin^2\theta\right).$$

4.2. *Three-dimensional analysis for axisymmetric bodies*

A significant difference between two- and three-dimensional flows is that the vorticity, which is first generated by a baroclinic torque, is then stretched and rotated by the $(\boldsymbol{\omega}_2\cdot\nabla)\boldsymbol{v}_1$, $(\boldsymbol{\omega}_2\cdot\nabla)\boldsymbol{v}_2$ terms in (2.14). In particular, when $\boldsymbol{U}\cdot\nabla\rho_0 = 0$, these terms generate a streamwise component of vorticity (see figure 5). The trailing vorticity may be calculated from the linearized vorticity equation (2.16):

$$\boldsymbol{\omega}_2(\boldsymbol{x}')\cdot\hat{\boldsymbol{s}} = \frac{v_1}{2\rho_B}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}\sin\alpha\int_{-\infty}^{s}\frac{U^2}{v_1^4}\frac{\partial v_1^2}{\partial n}\frac{(-2\psi_1/U)^{1/2}}{R}\mathrm{d}s + O\left(U\epsilon^2/a\right). \qquad (4.7)$$

The streamwise component of vorticity is (see Appendix B)

$$\boldsymbol{\omega}_2(\boldsymbol{x}')\cdot\hat{\boldsymbol{s}} = -\frac{U}{2\rho_B}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}\sin\alpha\frac{(-2\psi_1/U)^{1/2}}{R}\left(\frac{\partial X_d}{\partial n} - \frac{2v_{1,R}}{v_1}\right). \tag{4.8}$$

As to be expected from Yih's (1959) result, this is identical to Lighthill's (1953, equation 52) expression for the vorticity distribution downstream of a body in a sheared flow. The trailing vorticity far downstream ($x' \to -\infty$) is

$$\boldsymbol{\omega}_2\cdot\hat{\boldsymbol{s}} = -\frac{U}{2\rho_B}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}\sin\alpha\lim_{x'\to\infty}\frac{\partial X_d}{\partial n}. \tag{4.9}$$

Note that the vorticity is singular on the surface on the body and downstream-attached streamlines but the velocity field is finite everywhere. On the wake streamline ($\boldsymbol{U}\cdot\boldsymbol{x}' \to -\infty$), the secondary flow, $\boldsymbol{v}_2^r$, satisfying (2.18) and (4.9) is (see Auton 1987, equation 6.6)

$$\frac{v_{2,y}^r(R,\alpha)}{(U\mathrm{d}\rho_0/\mathrm{d}y)/2\rho_B} = -\frac{\sin 2\alpha}{R^2}\int_0^R X_d R\mathrm{d}R - \tfrac{1}{2}(1-\sin 2\alpha)X_d \tag{4.10}$$

$$\frac{v_{2,z}^r(R,\alpha)}{(U\mathrm{d}\rho_0/\mathrm{d}y)/2\rho_B} = -\frac{\cos 2\alpha}{R^2}\int_0^R X_d R\mathrm{d}R + \tfrac{1}{2}\cos 2\alpha X_d, \tag{4.11}$$

where $R = (y^2+x^2)^{1/2}$. The flow far from the centreline $R/a \gg 1$ is (Lighthill 1956, corrigendum)

$$\boldsymbol{v}_2^r(\boldsymbol{x}') = \frac{U}{\rho_B}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}\sin\alpha\frac{C_M\mathscr{V}}{8\pi}\nabla\left(\frac{1}{R}\left(1-\frac{x'}{r}\right)\right). \tag{4.12}$$

Equations (4.10), (4.11) and (4.12) are required to estimate the force on the body (Auton 1987). Figure 5 shows that the trailing vorticity gives a downthrust to the flow resulting in a lift force which drives the body towards the denser fluid. It also explains how the vorticity is generated by the finite baroclinic torque and is then amplified to a singular extent by the flow around the bluff body.

5. Flow when the body moves parallel to the density gradient, $\boldsymbol{U}\times\nabla\rho_0 = 0$

5.1. *Two-dimensional analysis*

In a two-dimensional flow, where there is no vortex stretching $(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} = \boldsymbol{0}$, the vorticity equation (2.14) reduces to

$$\frac{\mathrm{D}'(\rho_1\boldsymbol{\omega}_2\cdot\hat{\boldsymbol{z}})}{\mathrm{D}'t} = \hat{\boldsymbol{z}}\cdot\frac{\mathrm{D}'\boldsymbol{v}_1}{\mathrm{D}'t}\times\nabla\rho_1. \tag{5.1}$$

The baroclinic torque is calculated by using Lighthill's drift function, τ, where $\nabla\tau = -v_1(\mathrm{d}\tau/\mathrm{d}\psi_1)^{1/2}\hat{\boldsymbol{n}} + (1/v_1)^{1/2}\hat{\boldsymbol{s}}$, and the identity

$$-\hat{\boldsymbol{z}}\cdot\nabla(\tfrac{1}{2}v_1^2)\times\nabla\left(U\tau\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\right) = -v_1\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\left(-\frac{\partial\tau}{\partial n}\frac{\partial v_1}{\partial s} + \frac{\partial\tau}{\partial s}\frac{\partial v_1}{\partial n}\right)$$

$$= -v_1\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\left(\frac{\partial(\frac{1}{2}v_1^2)}{\partial s}\frac{\mathrm{d}\tau}{\mathrm{d}\psi_1} + \frac{1}{v_1}\frac{\partial v_1}{\partial n}\right) = v_1\frac{\partial}{\partial s}\left(\frac{1}{2}v_1\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{\partial\tau}{\partial n}\right).$$

By integrating the baroclinic torque along the base streamlines, it follows that the vorticity is

$$\boldsymbol{\omega}_2(\boldsymbol{x}')\cdot\hat{\boldsymbol{z}} = \frac{v_1}{2\rho_1}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{\partial\tau}{\partial n} = \frac{v_1}{2\rho_1}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\left(\frac{\partial X_d}{\partial n} - \frac{2v_{1,y}}{v_1}\right) = \frac{1}{\rho_1}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\left(\frac{1}{2}v_1\frac{\partial X_d}{\partial n} - v_{1,y}\right). \quad (5.2)$$

From (5.2) we see that to leading order the vorticity field is steady to $O(\epsilon)$ but increases/decreases at $O(\epsilon^2)$ depending on whether the local density is decreasing/increasing. Close to the attached streamline and to the surface S_B, the vorticity is $O(\epsilon U/n)$ and singular.

The rotational component of the secondary flow is calculated by writing $\boldsymbol{v}_2^r(\boldsymbol{x}') = \boldsymbol{v}_{2,a}^r(\boldsymbol{x}') + \boldsymbol{v}_{2,b}^r(\boldsymbol{x}')$, where $\boldsymbol{v}_{2,a}^r$, $\boldsymbol{v}_{2,b}^r$ are respectively the velocity fields associated with the downstream vorticity distribution and a localized vorticity distribution:

$$\hat{\boldsymbol{z}}\cdot\nabla\times\boldsymbol{v}_{2,a}^r = \frac{v_1}{2\rho_1}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{\partial X_d}{\partial n}, \quad \hat{\boldsymbol{z}}\cdot\nabla\times\boldsymbol{v}_{2,b}^r(\boldsymbol{x}') = -\frac{1}{\rho_1}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}v_{1,y}. \quad (5.3)$$

We determine the velocity field within a distance a/ϵ. In the far field, $\boldsymbol{v}_{2,a}^r(\boldsymbol{x}')$ is determined from the vorticity distribution

$$\omega_\infty(\boldsymbol{x}') = \begin{cases} \dfrac{U}{2\rho_B}\dfrac{\mathrm{d}\rho_0}{\mathrm{d}x}\lim\limits_{x'\to-\infty}\dfrac{\partial X_d}{\partial y}, & x' < 0, \\ 0, & x' > 0. \end{cases} \quad (5.4)$$

Because $\int\int(\omega_\infty - \hat{\boldsymbol{z}}\cdot\nabla\times\boldsymbol{v}_{2,a}^r)\mathrm{d}x\mathrm{d}y \to 0$ as the region of integration increases, the rotational component of the flow, $\boldsymbol{v}_{2,a}^r$, is solely determined by ω_∞ in the far field. Using the Biot-Savart law, it can be shown that far from the centreline, the secondary flow is

$$\boldsymbol{v}_{2,a}^r = \int_0^\infty\int_{-\infty}^\infty \omega_\infty\hat{\boldsymbol{z}}\times\left\{\frac{(x-\hat{x}, y-\hat{y})}{(x-\hat{x})^2+(y-\hat{y})^2} - \frac{(x-\hat{x}, y+\hat{y})}{(x-\hat{x})^2+(y+\hat{y})^2}\right\}\mathrm{d}\hat{x}\mathrm{d}\hat{y}$$
$$= \frac{\boldsymbol{x}'}{\rho_B|\boldsymbol{x}'|^2}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\int_0^\infty\frac{\partial X_d}{\partial\hat{y}}\hat{y}\mathrm{d}\hat{y} = -\frac{C_M\mathscr{V}U\boldsymbol{x}'}{\rho_B|\boldsymbol{x}'|^2}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}.$$

The above equation describes a flow consisting of a sink of strength $C_M\mathscr{V}(\boldsymbol{U}\cdot\nabla\rho_0)/2\rho_B$, located at the origin. The contribution to the flow from $\boldsymbol{v}_{2,b}^r$ is (from (5.2))

$$\boldsymbol{v}_{2,b}^r = -\frac{(C_M+1)\mathscr{V}}{4\pi\rho_B}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\cos 2\theta\frac{\boldsymbol{x}'}{|\boldsymbol{x}'|^2}. \quad (5.5)$$

From (5.3) the velocity field downstream of the body and near the centreline is

$$\lim_{x'\to-\infty} v_{2,x}^r(x,y) = -\frac{U}{2\rho_B}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}X_d, \qquad (y \neq 0) \quad (5.6)$$

which describes a localized jet-like flow along the centreline. Close to the centreline the secondary flow is $v_{2,x}^r = O(U\epsilon\log(a/n))$, where n is the distance from the attached streamline or body surface. As the body moves into denser fluid, the jet drives a flow in the opposite direction to $\boldsymbol{U}$ and the far-field flow is characterized by a sink located at the origin. The direction of the jet and sink is reversed when the body moves into less-dense fluid.

The pressure field is calculated from the steady Eulers' equation:

$$\frac{\partial}{\partial s}\left(p + \tfrac{1}{2}\rho(\boldsymbol{v}^2 - U^2)\right) = \frac{1}{2}\left(v - \frac{U^2}{v}\right)\boldsymbol{v}\cdot\nabla\rho. \quad (5.7)$$

The barotropic Bernoulli's equation is clearly not satisfied because the density along the streamline is not a function of p. The pressure variation along the streamlines is obtained by integrating (5.7) and (2.12c), to gives

$$p = p_0 + \tfrac{1}{2}\rho(v^2 - U^2) + \tfrac{1}{2}U\frac{\mathrm{d}\rho_0}{\mathrm{d}x}(UX_d - 2\phi_1) + O(\rho_B U^2\epsilon^2). \tag{5.8}$$

The pressure difference between two points upstream and downstream of the body is

$$[p]_{x'/a \gg 1}^{-x'/a \gg 1} = \rho_0 U [v_{2,x}]_{x'/a \gg 1}^{-x'/a \gg 1} + \frac{1}{2}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}U^2[X_d]_{x'/a \gg 1}^{-x'/a \gg 1} = O(\rho_B U^2\epsilon^2), \tag{5.9}$$

where we have used (5.6) to evaluate $v_{2,x}$ far downstream.

5.2. *Three-dimensional analysis for axisymmetric flow*

When the flow is axisymmetric, the only non-zero component of vorticity is in the azimuthal direction, so that the vorticity field may be regarded as an assemblage of vortex rings. As a vortex ring is advected past the body it is stretched, increasing the vorticity of the ring but conserving circulation. The generation of vorticity by stretching is $\hat{\boldsymbol{\varphi}}\cdot(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} = v_R\boldsymbol{\omega}\cdot\hat{\boldsymbol{\varphi}}/R$, where $\hat{\boldsymbol{\varphi}}$ is the unit vector in the azimuthal direction and v_R is the velocity component perpendicular to the centreline. The vorticity generation is described by

$$\frac{\mathrm{D}'}{\mathrm{D}'t}\left(\frac{\rho_1\boldsymbol{\omega}_2\cdot\hat{\boldsymbol{\varphi}}}{R}\right) = \frac{\hat{\boldsymbol{\varphi}}}{R}\cdot\frac{\mathrm{D}'\boldsymbol{v}_1}{\mathrm{D}'t}\times\nabla\rho_1. \tag{5.10}$$

The right-hand-side of (5.10) can be manipulated as

$$\begin{aligned}
-\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{\hat{\boldsymbol{\varphi}}}{R}\cdot\frac{\mathrm{D}'\boldsymbol{v}_1}{\mathrm{D}'t}\times\nabla\tau &= \frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{v_1}{R}\left(\frac{\partial\tau}{\partial n}\frac{\partial v_1}{\partial s} - \frac{\partial\tau}{\partial s}\frac{\partial v_1}{\partial n}\right)\\
&= \frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{v_1}{R}\left(\frac{v_1 R}{U(-2\psi_1/U)^{1/2}}\frac{\partial v_1}{\partial s}\frac{\mathrm{d}\tau}{\mathrm{d}(-2\psi_1/U)^{1/2}} - \frac{1}{v_1}\frac{\partial v_1}{\partial n}\right)\\
&= \frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{v_1}{U(-2\psi_1/U)^{1/2}}\frac{\partial}{\partial s}\left(\frac{1}{2}v_1^2\frac{\mathrm{d}\tau}{\mathrm{d}(-2\psi_1/U)^{1/2}}\right)\\
&\quad -\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{\frac{1}{2}v_1^3}{U(-2\psi_1/U)^{1/2}}\frac{\partial}{\partial s}\left(\frac{\mathrm{d}\tau}{\mathrm{d}(-2\psi_1/U)^{1/2}}\right) - \frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{1}{R}\frac{\partial v_1}{\partial n}\\
&= \frac{\mathrm{d}\rho_0}{\mathrm{d}x}\frac{v_1}{(-2\psi_1/U)^{1/2}}\frac{\partial}{\partial s}\left(\frac{1}{2}v_1^2\frac{\mathrm{d}\tau}{\mathrm{d}(-2\psi_1/U)^{1/2}}\right).
\end{aligned}$$

Integrating the above equation along the base flow streamlines shows that the vorticity distribution is

$$\boldsymbol{\omega}_2(\boldsymbol{x}')\cdot\hat{\boldsymbol{\varphi}} = \frac{1}{\rho_1}\frac{\mathrm{d}\rho_0}{\mathrm{d}x}\left(\frac{1}{2}v_1\frac{\partial X_d}{\partial n} - v_{1,R}\right). \tag{5.11}$$

Qualitatively the secondary flow generated by three-dimensional bodies is identical to that described in §5.1 and consists of a sink strength $C_M\mathscr{V}(\boldsymbol{U}\cdot\nabla\rho_0)/2\rho_B$ located at the origin, and a downstream 'jet' away from the body. The sign of the vorticity and direction of the wake flow change when $\mathrm{d}\rho_0/\mathrm{d}x < 0$. The analysis for the pressure variation along the unperturbed streamlines is identical to the two-dimensional flow, so that the pressure drop across the two ends of a streamline is $O(\rho_B\epsilon^2U^2)$. Figure 6 shows schematic diagram of the flow and vortex dynamics.

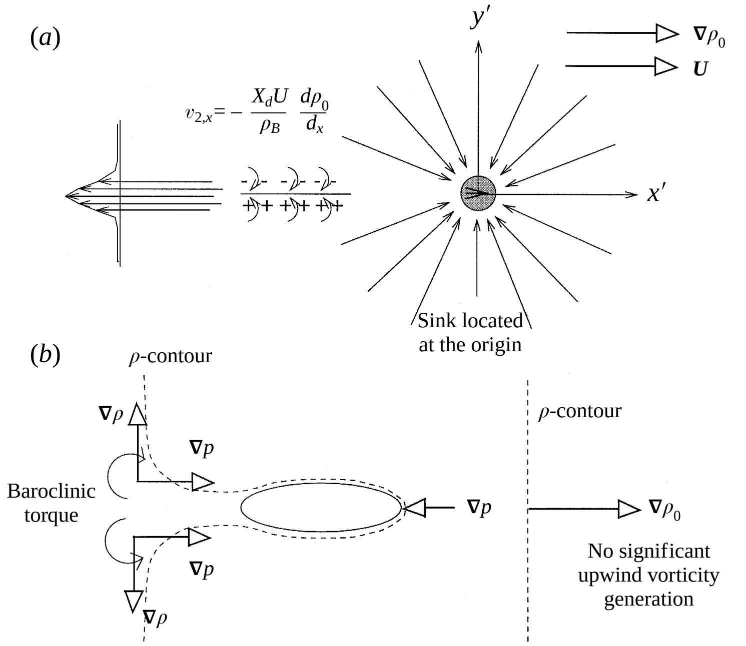

FIGURE 6. Flow around a two- or three-dimensional axisymmetric body moving parallel to the density gradient: (*a*) shows the strength of the sink at the origin $(C_M\mathscr{V}(\mathbf{U}\cdot\nabla\rho_0)/2\rho_B)$ and the downstream distribution of vorticity which gives rise to a flux of fluid away from the body, and (*b*) shows how the vorticity is generated immediately downstream of the body by the baroclinic couple.

Figure 7 shows the variation of the baroclinic torque along a streamline and in particular shows the separate contribution to the torque from the density gradient parallel and perpendicular to the base streamline. The vorticity distribution along the streamline is also shown.

6. Force on a body moving in a uniform density gradient

The force on a body moving in an inviscid fluid results from the pressure variation over the body surface, S_B, and is

$$\boldsymbol{F} = \int_{S_B} p\hat{\boldsymbol{n}}\mathrm{d}S \tag{6.1}$$

(Batchelor 1967, p. 138). The momentum-integral theorem is applied to the steady flow in a control volume $\mathscr{V}_\infty$ surrounding the body. The force on the body is independent of the shape of the control volume and may be chosen arbitrarily – in two dimensions we choose rectangle of length L and width W, and in three dimensions, a cylinder of radius W, where $1 \ll W/a \ll L/a \ll 1/\epsilon$. Within the region $\mathscr{D}_M$, the secondary flow, $\boldsymbol{v}_2$, is a valid approximation to the flow. Using Gauss's Theorem and the kinematic condition of the surface of the body, (6.1) can be manipulated to show

$$\boldsymbol{F} = -\int_{A\cup S} p\hat{\boldsymbol{n}}\mathrm{d}S - \int_{A\cup S} \rho(\boldsymbol{v}\cdot\hat{\boldsymbol{n}})\boldsymbol{v}\mathrm{d}S + \int_{\mathscr{V}_\infty} (\boldsymbol{v}\cdot\nabla\rho)\boldsymbol{v}\mathrm{d}V, \tag{6.2}$$

where A and S are respectively the end and the curved sides to the control volume. Note that the density field ρ is non-uniform in (6.2). The third term on the right-

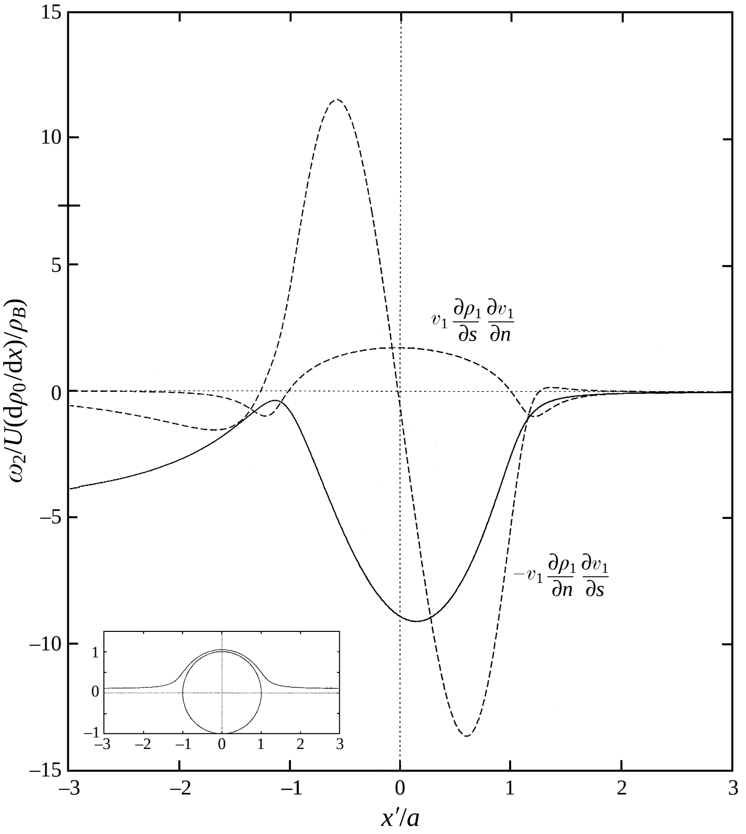

FIGURE 7. The contributions to $\mathrm{d}\omega_2/\mathrm{d}t$ from components of the baroclinic torque arising from the density gradient parallel and perpendicular to the streamlines along a single streamline $Y = 0.1a$, using for illustration the flow around a cylinder. The streamline corresponding to $Y = 0.1a$ is plotted in the inset. The variation of the vorticity along the streamline is shown as the full line.

hand-side of (6.2) is zero when $\boldsymbol{U}\cdot\nabla\rho_0 = 0$, but it makes an important contribution when the body moves parallel to the density gradient.

6.1. *Force on a body moving perpendicularly to the density gradient,* $\boldsymbol{U}\cdot\nabla\rho_0 = 0$

The force on a body moving perpendicularly to a density gradient can be deduced from the force on a body moving in a shear defined as $\mathrm{d}U_\infty/\mathrm{d}y$. Batchelor (1967) showed that the lift coefficient describing the lift force acting on a two-dimensional body is

$$\boldsymbol{F} = \rho_B C_L \mathscr{V} \frac{\mathrm{d}U_\infty^2}{\mathrm{d}y}\hat{\boldsymbol{y}}, \tag{6.3}$$

where $C_L = (C_M + 1)/2$. Auton (1987) calculated the lift force on a sphere fixed in a shear, and Wallis (1996) generalized these results to bodies axisymmetric about the direction of relative motion and showed that they are subject to the lift force (6.3), where the lift coefficient is $C_L = C_M/2$. In both cases the lift force tends to push the bodies towards the faster flow. Bernoulli's equation is satisfied along the streamline when the body moves perpendicularly to a density gradient or is fixed in a shear and the pressure fields in both flows are identical, although the velocity fields differ. Thus the force on a two-dimensional or axisymmetric body moving perpendicularly to a density gradient is given by (6.3) for a shear where $\mathrm{d}U_\infty/\mathrm{d}y = U(\mathrm{d}\rho_0/\mathrm{d}y)/2\rho_B$, i.e.

$$\boldsymbol{F} = C_L \mathscr{V} U^2 \frac{\mathrm{d}\rho_0}{\mathrm{d}y}\hat{\boldsymbol{y}}. \tag{6.4}$$

The total lift force acting on a body moving in a shear and perpendicular to a density gradient is a linear combination of (6.3) and (6.4). The expression (6.4) for the force for two- and three-dimensional axisymmetric bodies can also be derived using the expressions for the far-field flow, because only certain terms in the expansion contribute to $\boldsymbol{F}$. The force was verified directly for the case of cylinder by integrating the pressure variation over the body surface.

When $\boldsymbol{U}\cdot\nabla\rho_0 = 0$, the lift force can be written vectorially as

$$\boldsymbol{F} = C_L \mathscr{V} (\boldsymbol{U} \times \nabla\rho_0) \times \boldsymbol{U}, \tag{6.5}$$

where the lift coefficient C_L is respectively $(C_M + 1)/2$ or $C_M/2$ for two- or three-dimensional bodies which are axisymmetric about the direction of motion. The direction of the lift force is unchanged when $\boldsymbol{U}$ is reversed.

6.2. *Force on a body moving parallel to density gradient,* $\boldsymbol{U} \times \nabla\rho_0 = \boldsymbol{0}$

The force on a two- or three-dimensional axisymmetric body moving parallel to the density gradient is calculated in Appendix C, where we show that

$$\boldsymbol{F} = -C_M \mathscr{V} U \boldsymbol{U} \frac{\mathrm{d}\rho_0}{\mathrm{d}x} + \rho_B \boldsymbol{U} \int_A \boldsymbol{v}_2^r \cdot \hat{\boldsymbol{n}}\, \mathrm{d}S. \tag{6.6}$$

Unlike barotropic flows, with baroclinic effects the drag can no longer be interpreted simply in terms of the momentum deficit of the flow downstream of the body. The first term describes the pressure drag and the momentum deficit caused by the distortion of the density field by the *irrotational* component of the flow. The second term describes the momentum deficit due to the downstream *velocity perturbation* caused by the *rotational* component of the flow, as in barotropic flows.

The downstream volume flux of the perturbed flow for a two- or three-dimensional body axisymmetric about $\boldsymbol{U}$ is calculated from (5.7) and Darwin's proposition:

$$\int_A \boldsymbol{v}_2^r \cdot \hat{\boldsymbol{n}} \mathrm{d}S = \frac{1}{2\rho_B} C_M \mathscr{V} U \frac{\mathrm{d}\rho_0}{\mathrm{d}x}. \tag{6.7}$$

As a result, from (6.6) and (6.7), we see that the changes in the momentum flux, because it is positive, exert a *thrust* on the body, as the body moves into denser fluid, whereas the density variation effectively exerts a drag on the body which is greater than the thrust. When $\boldsymbol{U} \times \nabla\rho_0 = \boldsymbol{0}$, the total force can be written vectorially as

$$\boldsymbol{F} = -C_D \mathscr{V} (\boldsymbol{U}\cdot\nabla\rho_0)\boldsymbol{U}, \tag{6.8}$$

where the drag coefficient $C_D = C_M/2$ for axisymmetric and two-dimensional bodies. The direction of the drag force is reversed when the body moves into lighter fluid.

6.3. *Force on a body moving with a constant velocity* $\boldsymbol{U}$ *in a uniform density gradient,* $\nabla\rho_0$

An axisymmetric body moving parallel to a density gradient generates only a vorticity component binormal to the base streamlines. In contrast, such a body moving perpendicularly to the density gradient generates components of vorticity normal, tangential and binormal to the base streamlines. The vorticity components parallel and normal to the streamlines are $O(|\boldsymbol{U} \times \nabla\rho_0|/\rho_0)$, but the binormal component consists of a linear combination of a localized contribution $O(|\boldsymbol{U} \times \nabla\rho_0|/\rho_0)$ and a non-localized contribution $O(\boldsymbol{U}\cdot\nabla\rho_0)/\rho_0$. Since the vorticity distribution is a linear combination of the effects due to a body moving parallel and perpendicularly to $\nabla\rho_0$, the secondary flow and hence the force acting on a body axisymmetric about $\boldsymbol{U}$ is

$$\boldsymbol{F} = -C_D\mathscr{V}(\boldsymbol{U}\cdot\nabla\rho_0)\boldsymbol{U} + C_L\mathscr{V}(\boldsymbol{U} \times \nabla\rho_0) \times \boldsymbol{U}, \tag{6.9}$$

where $C_L, C_D > 0$ are respectively lift and drag coefficients.

The vorticity field generated by a two-dimensional body moving with a constant velocity $\boldsymbol{U}$ at an arbitrary angle to a uniform density gradient $\nabla\rho_0$ is a linear combination of the vorticity generated by moving parallel or perpendicularly to the density field because there is no vortex stretching. It follows that the total force on a two-dimensional body is also described by (6.9).

6.4. *Force on a body moving unsteadily in a uniform density field*

We have established that a body moving steadily in a weak density gradient is subject to the force described by (6.9) for time $|\boldsymbol{U}|t/a \gg 1$. The flow past the body is quasi-steady providing the time scale for velocity variations, $|\boldsymbol{U}|/|\dot{\boldsymbol{U}}|$, is much larger than the advective time scale, $a/|\boldsymbol{U}|$. This is clearly satisfied by (6.9) because $a|\dot{\boldsymbol{U}}|/U^2 = O(\epsilon)$, so that the force on a body moving unsteadily in a uniform density gradient is also described by (6.9).

The lift and drag coefficients are functions of the added-mass coefficient C_M, which depends on the orientation of the body (3.5), so that the lift and drag coefficients may be unsteady when $\boldsymbol{U}/|\boldsymbol{U}|$ is unsteady. For the specific case of a cylinder and sphere, the lift and drag coefficients are independent of orientation.

7. The movement of a cylinder or sphere in a simple non-uniform density field

From the general expression of the force on a body (6.9), we can now calculate the trajectory of a cylinder or sphere located at $(x(t), y(t))$ moving at a velocity (u_x, u_y) in the (x, y)-plane in an inviscid fluid with a positive density gradient $(\mathrm{d}\rho_0/\mathrm{d}y)\hat{\boldsymbol{y}}$. When the flow has a mean velocity $\overline{\boldsymbol{U}}$, the velocity of the body should be expressed relative to $\overline{\boldsymbol{U}}$ in order to interpret the following results.

The body is projected with an initial velocity $(u_x^{(p)}, u_y^{(p)})$ from the origin. The equations of motion for a point-symmetric body of density ρ_b acted upon by a combination of the added-mass force, $-\rho_0(y)C_M\mathscr{V}\mathrm{d}\boldsymbol{U}/\mathrm{d}t$, and the fluid force (6.9), is

$$\frac{\mathrm{d}u_x}{\mathrm{d}t} = -(\tilde{C}_D + \tilde{C}_L)u_x u_y/a, \tag{7.1}$$

$$\frac{\mathrm{d}u_y}{\mathrm{d}t} = (\tilde{C}_L u_x^2 - \tilde{C}_D u_y^2)/a, \tag{7.2}$$

where $\tilde{C}_D, \tilde{C}_L = a(C_D, C_L)|\nabla\rho_0|/(\rho_b + \rho_0(y)C_M)$. These coefficients (which are small) can be assumed to be constants providing the normal displacement is less than the distance over which the density changes significantly, i.e. $y(t)(\mathrm{d}\rho_0/\mathrm{d}y)/\rho_0 \ll 1$. Since the forces are weak, the curvature of the trajectory is large and (7.1), (7.2) are valid. Equation (7.2) shows that all bodies projected horizontally (i.e. $u_y^{(p)} \ll u_x^{(p)}$) are subject to a lift force, tending to push them towards the denser fluid, even if they are initially projected towards the lighter fluid and are initially accelerated. Figure 8 shows examples of trajectories of particles projected at an angle α to the horizontal.

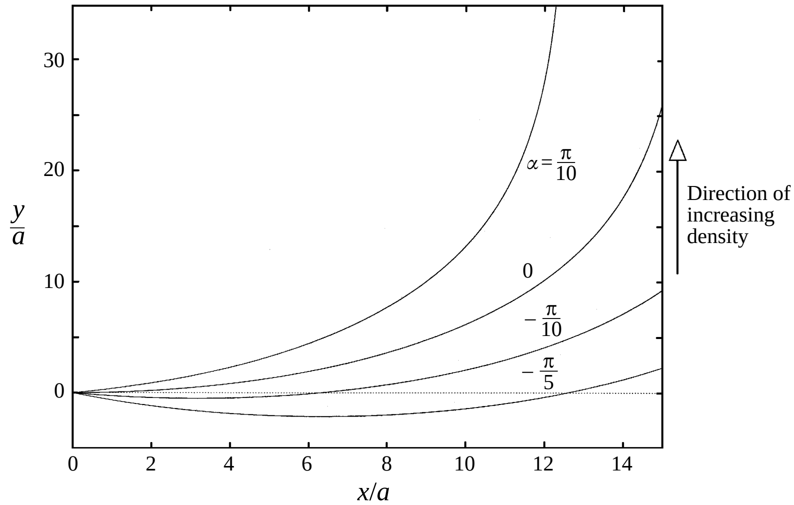

FIGURE 8. Typical trajectories of bodies with density ρ_p, which are projected in a fluid of density ρ_0 and gradient $\nabla\rho_0$, with unit speed at an angle α to the x-axis. The normalized drag and lift coefficients are chosen as $\tilde{C}_D = 1/10a$, $\tilde{C}_L = 1/10a$, where $\tilde{C}_D, \tilde{C}_L = (C_D, C_L)|\nabla\rho_0|/(\rho_0 C_M + \rho_b)$. The time difference increment between the positions is $a/10U$; $\alpha = -\pi/5$, $-\pi/10$, 0, $\pi/10$.

Equations (7.1), (7.2) also show that as u_y increases, u_x decreases until at $t = t_{max}$, $\tilde{C}_L u_x^2 = \tilde{C}_D u_y^2$. Thereafter, for $t \gg t_{max}$, u_y decreases slowly ($u_y \sim a/(\tilde{C}_D t)$), while u_x decreases more rapidly as $u_x \sim U(Ut/a)^{-2}$ for both cylinders and spheres.

Thus whatever the initial direction of the trajectories (other than exactly antiparallel to the density gradient), the body eventually travels parallel to the density gradient, and the velocity decreases over a distance $a/\tilde{C}_D$. For bodies projected at a steep angle α greater than $\tan^{-1}(\tilde{C}_D/\tilde{C}_L)^{1/2}$, u_y decreases immediately (for $t > 0$) and the drag force inhibits motion against the density gradient. A body projected towards the lower-density fluid accelerates in both the x- and y-directions, but u_x increases at a faster rate than u_y and eventually, at a distance $|y|$ of $O(\rho_0/|\nabla\rho_0|)$, the body's motion is reversed, as indicated in figure 8 for $\alpha < 0$, and it starts to move towards the denser fluid.

8. Concluding remarks

In this paper we have calculated the first-order steady flow generated by a body moving in a density gradient. The first-order steady-state solutions developed here are not valid in the far field ($r \gg a/\epsilon$), nor in the inner region $\mathscr{D}_I$. The singularities associated with the density gradient and vorticity fields in $\mathscr{D}_I$ are a general feature of inviscid flows past impermeable bodies and result from the stretching of isopycnal surfaces and vortical elements near the stagnation points. When the bodies are cusped at the attached streamlines, the density gradient and vorticity field are not singular and the expressions for the velocity field are valid on the body surface providing $\epsilon \ll 1/\log(a/b)$, where b is the radius of curvature of the cusp. Similarily, the effect of molecular diffusion will be to smooth out large density gradients, reducing the baroclinic torque in $\mathscr{D}_I$, and when $\boldsymbol{U} \times \nabla\rho_0 = \boldsymbol{0}$, the vorticity field will be finite on the body surface.

The main results of this paper are the following.

(i) The flow around two-dimensional bodies moving perpendicularly to a density gradient is affected by the density gradient in the locality of the body and does not

Geometry	$\nabla\rho_0 \cdot \boldsymbol{U} = 0$ $\rho_0 = \rho_0(y);\quad \boldsymbol{U} = (U, 0, 0)$ Lift force	$\nabla\rho_0 \times \boldsymbol{U} = \boldsymbol{0}$ $\rho_0 = \rho_0(x);\quad \boldsymbol{U} = (U, 0, 0)$ Drag force
2D body	$\frac{1}{2}(C_M + 1)\mathscr{V}U^2\frac{\mathrm{d}\rho_0}{\mathrm{d}y}$	$-\frac{1}{2}C_M\mathscr{V}U^2\frac{\mathrm{d}\rho_0}{\mathrm{d}x}$
3D axisymmetric body	$\frac{1}{2}C_M\mathscr{V}U^2\frac{\mathrm{d}\rho_0}{\mathrm{d}y}$	$-\frac{1}{2}C_M\mathscr{V}U^2\frac{\mathrm{d}\rho_0}{\mathrm{d}x}$

TABLE 1. Summary of results for the force on a body moving in a constant density gradient

produce any perturbation far from the body either by a downstream vorticity wake (in the manner of three-dimensional bodies) or by means of waves in the far field (in the manner of stably stratified flow). The flow produces a lift towards the denser fluid whatever the relative velocity of the body and the fluid.

(ii) A three-dimensional body, axisymmetric about $\boldsymbol{U}$, experiences a lift force but induces a trailing vorticity component parallel and antiparallel to its velocity $\boldsymbol{U}$, consequently generating a perturbation swirl velocity $O(a|\nabla\rho_0 \times \boldsymbol{U}|)$, which extends a distance $|\boldsymbol{U}|t$ downstream. We also established a relationship between the force acting on a body moving perpendicularly to a density gradient or in a weakly sheared flow (see Appendix A and table 1). This suggests that the experimental results of Wallis (1996) for the force on bodies fixed in a shear also give information about the lift force on a body moving perpendicularly to a density gradient. In addition, the results from studies of bodies moving in a weak shear, such as the Pitot displacement effect (Lighthill 1957), may be relevant to bodies moving in variable-density flows.

(iii) When a two-dimensional or axisymmetric body moves parallel to the density gradient it experiences a force opposing its motion as it moves towards denser fluid and accelerating its motion as it moves towards lighter fluid (see table 1). This is qualitatively different to the effect of buoyancy forces in a weak density gradient which leads to a drag force which does not depend on whether the body moves up or down because fluid increases its potential energy when displaced in either direction (Warren 1960).

(iv) Finally, the movement of a volume parallel to a density gradient produces a remarkable inviscid wake 'jet' extending a distance Ut downstream. The direction of the jet depends on whether the body moves into denser or lighter fluid. An estimate of the jet speed is $v_{2,x} \sim -(\boldsymbol{U}\cdot\nabla\rho_0)|\ln(n/a)|a/\rho_0$, where n is the distance from the attached streamline, and n is typically of the order of the boundary layer thickness in viscous flows. When bodies are propelled downwards through an air/water interface they produce an energetic upward spray jet probably aided by this mechanism.

Applying the lift and drag force formulae to trajectories of cylinder or spheres shows that since both the inertial force and the rate of change of momentum are both proportional to the volume $\mathscr{V}$, the trajectories are independent of $\mathscr{V}$. Also, bodies projected with a velocity $\boldsymbol{U}$ in a region with a density $\nabla\rho_0$ initially slow down or accelerate over a distance $\rho_0/|\nabla\rho_0|$, depending on the sign of $(\boldsymbol{U}\cdot\nabla\rho_0)$ and the relative magnitude of $|\boldsymbol{U} \times \nabla\rho_0|$, but ultimately tend to move in a direction parallel to the density gradient whatever their initial direction.

These results can be used to infer the motion of small-scale eddies or fluid lumps (Prandtl's 1925, 'flussigkeit ballen') in turbulent flows with strong density gradient by considering them as the closed moving volume around vortex rings. They suggest that fluid lumps moving relative to the mean flow and perpendicularly to a density gradient would tend to be forced perpendicular to the flow and towards higher density. This hypothesis is consistent with Panchapakesan & Lumley's (1993) study of low-density jets: they observed that the spreading rate and entrainment are enhanced in low-density jets surrounded by fluid with higher density (in this case $\rho_0/|\nabla\rho_0|$ is much greater than the jet width). However, this conclusion does not apply to low-density shear layers (Hermanson & Dimotakis 1989), where the flow is dominated by large two-dimensional coherent structures. In this particular case the coherent structures composed of low-density fluid are stabilized, perhaps by a similar action to that of bubbles being attracted to a vortex centre (Sene, Hunt & Thomas 1994), thereby reducing entrainment and spreading rate. Clearly further research is required to reconcile the differences between observations in mixing layers and jets.

In the absence of shear, a mean density gradient also affects the diffusion of turbulence and the boundary entrainment velocity E_b (Turner 1973) by imposing a net force on a fluid element moving parallel to the density gradient. Thus turbulent diffusion and entrainment may be increased or decreased depending on whether the density gradient is antiparallel or parallel to E_b. These effects are probably even more important in flames where the sharp density gradients at reaction fronts perhaps reduce turbulent entrainment because in this case $\rho_0/|\nabla\rho_0|$ is comparable with the thickness of the flame region.

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Appendix A

The connection between the flow around a body moving steadily perpendicularly to a density gradient and in a sheared flow is now explored. When the unperturbed density field $\rho_0(y)$ has a rectilinear gradient and $\boldsymbol{U} = (U, 0, 0)$, the relative velocity field $\boldsymbol{v}$ is steady and the density is constant along the streamlines (in the frame of the body). The velocity field, $\boldsymbol{v}$, can be transformed by Yih's (1959) result (1.1) to the steady flow around a body moving in a shear (see Drazin's 1961 discussion). Since the momentum and continuity equation can be written in streamline coordinates

$$\rho v_s \frac{\partial v_s}{\partial s} = -\frac{\partial p}{\partial s} \qquad \text{and} \qquad \frac{\partial(v_s A)}{\partial s} = 0, \tag{A 1}$$

where A is cross-section of a streamline and may be rewritten as

$$(\rho/\rho_B)^{1/2}\frac{\partial}{\partial s}((\rho/\rho_B)^{1/2}v_s) = -\frac{\partial p}{\partial s},$$

because $\partial\rho/\partial s = 0$ and $\partial(\rho^{1/2}v_s A)/\partial s = 0$, and ρ_B is a reference density. Defining $v_s' = v_s(\rho/\rho_0)^{1/2}$ shows that the rescaled velocity $\boldsymbol{v}'$ satisfies

$$\nabla\cdot\boldsymbol{v}' = 0, \qquad \rho_0(\boldsymbol{v}'\cdot\nabla)\boldsymbol{v}' = -\nabla p, \tag{A 2}$$

subject to the boundary conditions $\boldsymbol{v}' \to -\boldsymbol{U}(\rho_0(y)/\rho_B)^{1/2}$ as $\boldsymbol{x}'\cdot\boldsymbol{U} \to \infty$. The rescaled velocity $\boldsymbol{v}'$ is equivalent to that in a constant-density flow around a body moving in a shear $U(\mathrm{d}\rho_0/\mathrm{d}y)/2\rho_B$ (Hawthorne & Martin 1955). By using the expression for the force on the body (6.1), and Bernoulli's barotropic equation, the force on a body moving in a shear described by (A2) is identical to the force on a body moving perpendicularly to a density gradient.

Appendix B

The equivalence between the Hawthorne & Martin (1955) solution for the flow generated by an axisymmetric body moving into a density gradient, and Lighthill's (1956) solution for the vorticity field generated by the same body moving in a weakly sheared flow is demonstrated. The gradient of the drift function perpendicular to the base streamlines is

$$\frac{\mathrm{d}\tau}{\mathrm{d}(-2\psi_1/U)^{1/2}} = -\int_{-\infty}^{\phi_1} \frac{1}{v_1^4} \frac{\mathrm{d}v_1^2}{\mathrm{d}(-2\psi_1/U)^{1/2}} \mathrm{d}\phi, \tag{B 1}$$

where τ is defined by (3.7). The gradient of the streamfunction is $\nabla\psi_1 = (v_r\hat{\boldsymbol{\theta}} - v_\theta\hat{\boldsymbol{r}})R$, and since $|\nabla(-2\psi_1/U)^{1/2}| = v_1 R/U(-2\psi_1/U)^{1/2}$, then

$$\frac{\mathrm{d}v_1^2}{\mathrm{d}(-2\psi_1/U)^{1/2}} = \frac{\nabla v_1^2 \cdot \nabla(2\psi/U)^{1/2}}{|\nabla(-2\psi_1/U)^{1/2}|^2} = \frac{U(-2\psi_1/U)^{1/2}}{v_1 R} \frac{\partial v_1^2}{\partial n}. \tag{B 2}$$

Hence,

$$\int_{-\infty}^{s} \frac{U}{v_1^3} \frac{(-2\psi_1/U)^{1/2}}{v_1 R} \frac{\partial v_1^2}{\partial n} \mathrm{d}s = -\frac{\mathrm{d}\tau}{\mathrm{d}(-2\psi_1/U)^{1/2}}. \tag{B 3}$$

Therefore

$$v_1 \int_{-\infty}^{s} \frac{U^2}{v_1^4} \frac{(-2\psi_1/U)^{1/2}}{R} \frac{\partial v_1^2}{\partial n} \mathrm{d}s = -\frac{U^2(-2\psi_1/U)^{1/2}}{R} \frac{\partial \tau}{\partial n} = \frac{U(-2\psi_1/U)^{1/2}}{R} \left(\frac{\partial X_d}{\partial n} - \frac{2v_{1,R}}{v_1} \right). \tag{B 4}$$

Appendix C

The force acting on a two- or three-dimensional axisymmetric body moving parallel to the density gradient is calculated from

$$F_x = \underbrace{-\int_A (p + \rho \boldsymbol{v}^2) n_x \mathrm{d}S}_{(1)} \underbrace{- \int_S \rho(\boldsymbol{v} \cdot \hat{\boldsymbol{n}}) v_x \mathrm{d}S}_{(2)} + \underbrace{\int_{\mathscr{V}_\infty} (\boldsymbol{v} \cdot \nabla\rho) v_x \mathrm{d}V}_{(3)}. \tag{C 1}$$

In two dimensions, $\mathscr{V}_\infty = LW$ and $\mathscr{V}_\infty = \pi L W^2$ in three dimensions.

Term (1)

From (5.9), the pressure drop across the two ends of the control volume is $O(\rho_B \epsilon^2 U^2)$, and is negligible. Therefore

$$\begin{aligned} \int_A (p + \rho \boldsymbol{v}^2) n_x \mathrm{d}S &= \int_A \rho(U^2 - 2U(v_{1,x} + U + v_{2,x})) n_x \mathrm{d}S \\ &= U^2 \int_A \rho_1 n_x \mathrm{d}S - 2 \int_A \rho U(v_{1,x} + U) n_x \mathrm{d}S - 2\rho_B U \int_A \boldsymbol{v}_2 \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S. \end{aligned} \tag{C 2}$$

The first term on the right-hand side of (C2) is

$$\int_A \rho_1(\boldsymbol{x}')n_x \mathrm{d}S = \int_A \left(\lim_{x'/a \gg 1} \rho_1(\boldsymbol{x}') - \lim_{-x'/a \gg 1} \rho_1(\boldsymbol{x}') \right) \mathrm{d}S = (\mathscr{V}_\infty - O(B\mathscr{V}) + C_M\mathscr{V}) \frac{\mathrm{d}\rho_0}{\mathrm{d}x}, \tag{C 3}$$

where $B = \mu W/\mathscr{V}L$ in two dimensions, and $B = \mu W^2/\mathscr{V}L^2$ in three dimensions. It should be noted that ρ_1 is singular on the centreline, but the integral of ρ_1 across A is not singular. The rotational component of the flow decays more slowly than the irrotational component, so that $|\nabla\phi_2| \ll |\boldsymbol{v}_2^r|$ on S and A. Therefore the contribution from Term (1) is

$$-(C_M\mathscr{V} + \mathscr{V}_\infty)U^2 \frac{\mathrm{d}\rho_0}{\mathrm{d}x} + 2\rho_B U \int_A \boldsymbol{v}_2^r \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S - O(\rho_B B\mathscr{V}U^2). \tag{C 4}$$

Term (*2*)

$$\int_S \rho(\boldsymbol{v}\cdot\hat{\boldsymbol{n}})v_x \mathrm{d}S = \int_S \rho(\boldsymbol{v}_1\cdot\hat{\boldsymbol{n}} + \boldsymbol{v}_2\cdot\hat{\boldsymbol{n}})(v_{1,x} + v_{2,x})\mathrm{d}S$$

$$= -\int_S \rho U \boldsymbol{v}_1\cdot\hat{\boldsymbol{n}} \, \mathrm{d}S + \int_S \rho(v_{1,x} + U)\boldsymbol{v}_2\cdot\hat{\boldsymbol{n}} \, \mathrm{d}S - \int_S \rho_B \boldsymbol{v}_2\cdot\hat{\boldsymbol{n}} U \, \mathrm{d}S + \int_S \rho_B \boldsymbol{v}_1\cdot\hat{\boldsymbol{n}} v_{2,x} \, \mathrm{d}S,$$

where terms of $O(\epsilon^2)$ have been ignored. The second and fourth terms are negligible compared to the first and third. The first term is

$$\int_S \left(\rho_B + x' \frac{\mathrm{d}\rho_0}{\mathrm{d}x} \right) \boldsymbol{v}_1\cdot\hat{\boldsymbol{n}} \, \mathrm{d}S = O(\rho_B B\mathscr{V}) + (C_M + 1)\mathscr{V}U \frac{\mathrm{d}\rho_0}{\mathrm{d}x}. \tag{C 5}$$

The fluid is incompressible so that

$$\int_S \boldsymbol{v}_2^r \cdot\hat{\boldsymbol{n}} \, \mathrm{d}S = -\int_A \boldsymbol{v}_2^r \cdot\hat{\boldsymbol{n}} \, \mathrm{d}S. \tag{C 6}$$

The contribution from Term (2) is

$$(1 + C_M)\mathscr{V}U^2 \frac{\mathrm{d}\rho_0}{\mathrm{d}y} - \rho_B U \int_A \boldsymbol{v}_2\cdot\hat{\boldsymbol{n}} \, \mathrm{d}A. \tag{C 7}$$

Term (*3*)

$$\int_{\mathscr{V}_\infty} (\boldsymbol{v}\cdot\nabla\rho)v_{1,x} \, \mathrm{d}V + \int_{\mathscr{V}_\infty} (\boldsymbol{v}\cdot\nabla\rho)v_{2,x} \, \mathrm{d}V.$$

The secondary velocity, $v_{2,x}$, has a logarithmic singular on S_B and an attached downstream streamline but the volume integral of $v_{2,x}$,

$$\left| \int_{\mathscr{V}_\infty} v_{2,x} \mathrm{d}V \right| = O(\epsilon U\mathscr{V}),$$

is finite. In addition,

$$\int_{\mathscr{V}_\infty} \boldsymbol{v}_1 \mathrm{d}V = \mathscr{V}_\infty U - (C_M + 1)\mathscr{V}U - O(B\mathscr{V}U).$$

Using (2.12*b*) and the above expression for the volume integrals, we find that the contribution from Term (3) is

$$-(\mathscr{V}_\infty - (C_M + 1)\mathscr{V} - O(B\mathscr{V})) \, U^2 \frac{\mathrm{d}\rho_0}{\mathrm{d}x}.$$

The total force is a given by the sum of Terms (1), (2) and (3). The force on the body is independent of the control volume. An alternative choice of the control volume could be the region bounded by the base streamlines; for this particular choice of control volume, there are two additional contributions because the volume, $\mathscr{V}_\infty$, is decreased and secondly $\boldsymbol{v}\cdot\hat{\boldsymbol{n}} \neq 0$ on S.

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